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Rough approximations of vague sets in fuzzy approximation space

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ABSTRACT

The combination of the rough set theory, vague set theory and fuzzy set theory is a novel research direction in dealing with incomplete and imprecise information. This paper mainly concerns the problem of how to construct rough approximations of a vague set in fuzzy approximation space. Firstly, the β-operator and its complement operator are introduced, and some new properties are examined. Secondly, the approximation operators are constructed based on β-(complement) operator. Meantime, λ-lower (upper) approximation is firstly proposed, and then some properties of two types of approximation operators are studied. Afterwards, for two different kinds of approximation operators, we introduce two roughness measure methods of the same vague set and discuss a property. Finally, an example is given to illustrate how to calculate the rough approximations and roughness measure of a vague set using the β-(complement) product between two fuzzy matrixes. The results show that the proposed rough approximations and roughness measure of a vague set in fuzzy environment are reasonable.

1. Introduction

Fuzzy set theory was first proposed by Zadeh [1]. It is an important mathematical approach to uncertain and vague data analysis, and has been widely used in the area of fuzzy decision making, fuzzy control, fuzzy inference, and so on [2–5]. Thereafter, the theory of rough set was proposed by Pawlak [6], which was though of as another powerful tool for managing uncertainty that arises from inexact, noisy, or incomplete information. In terms of method, it was turned out to be methodologically significant in the domains of artificial intelligence and cognitive science, especially when representating or reasoning with imprecise knowledge, machine learning and knowledge discovery. In recent years, the combination of fuzzy set theory and rough set theory has been studied by many researchers [7–13,31–35], hence, many new mathematical methods are generated for dealing with the uncertain and imprecise information, such as the fuzzy rough sets and rough fuzzy sets, etc. Meantime, many metric methods are presented and investigated by different authors, in order to measure the uncertainty and ambiguity of the different sets [14–20].

More recently, the notion of vague sets was introduced by Gau and Buehrer [21]. In fact, it was the same concept with the intuitionistic fuzzy sets proposed by Atanassov et al. [22,28]. In the last two decades, many scholars have been interested in the theory and already made further studies [23–28]. Nowadays, there have been many applications of vague sets in medical diagnosis and decision making, etc. The integration of vague set theory and rough set theory has been done in 2005 [29]. The rough approximations of a vague set in Pawlak approximation space were presented and the roughness measure of a vague set had emerged. Subsequently, the concept of rough vague sets was provided by Al-Rababah [30]. However, the main results of the previous two papers are limited in the Pawlak approximation space based on the equivalence relation. Consequently,

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we take into consideration the combination of the vague set theory, rough set theory and fuzzy set theory. Here, we not only to define the rough approximations of a vague set in fuzzy approximation space, but also give roughness measure of a vague set in fuzzy approximation space.

The rest of this paper falls into five parts. In Section 2, we review the basic concepts of the vague sets, β-operator and the fuzzy approximation space, etc. Meanwhile, we quote and prove several new properties of the β-operator. And the concept of β-complement operator is introduced. This establishes a basis for the definition of the approximation operators of the vague sets in fuzzy approximation space and the discussion of their properties partially. In Section 3, we construct the lower and upper approximations of a vague set in fuzzy approximation space based on the β-operator and β-complement operator. Simultaneously, we prove that the expressions of approximation operators meet the demand of vague sets. Following, we bring forward the concepts of the λ-lower and λ-upper approximations of a vague set using the λ-cut relation on fuzzy approximation space. Later on several properties of two types of approximation operators are studied. In Section 4, two roughness measures of a vague set for the different approximation operators are posed, and a corresponding property is examined. In Section 5, an example is given to illustrate the proposed concepts. Finally, we conclude in Section 6.

2. Preliminaries

For completeness and clarity, we introduce some basic knowledge and notions of vague set theory and fuzzy set theory in this section. In addition, the β-operator and β-complement operator are introduced to elicit the concepts of lower and upper approximations of a vague set in fuzzy approximation space, and some properties of β-operator are discussed here.

2.1. Vague sets

In this part, we review some related concepts of vague sets introduced by Gau and Buehrer [21].

Let $U = \{u_1, u_2, \ldots, u_n\}$ be the universe of discourse, u_i denotes a generic element of U. A vague set A in the universe of discourse U is characterized by a truth-membership function t_A and false-membership function f_A given by

$$t_A : U \to [0,1],$$
$$f_A : U \to [0,1],$$

where $t_A(u_i)$ is a lower bound on the grade of membership of u_i derived from "the evidence for u_i", $f_A(u_i)$ is a lower bound on the negation of u_i derived from "the evidence against u_i", and $t_A(u_i) + f_A(u_i) \leqslant 1$. Thus the grade of membership of u_i in the vague set A is bounded to a subinterval $[t_A(u_i), 1 - f_A(u_i)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u_i)$, then $t_A(u_i) \leqslant \mu(u_i) \leqslant 1 - f_A(u_i)$.

In general, the vague set A is written as $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in U\}$, where the interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A.

Next, we will give some rules of operations containing equality, inclusion, intersection, union and complement of the vague sets.

Definition 2.1. Let $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in U\}$ and $B = \{\langle x, t_B(x), f_B(x)\rangle : x \in U\}$ be two vague sets of the universe of discourse U, then

(a) *equality:* $A = B$ iff $\forall x \in U, t_A(x) = t_B(x)$ and $f_A(x) = f_B(x)$;
(b) *inclusion:* $A \subseteq B$ iff $\forall x \in U, t_A(x) \leqslant t_B(x)$ and $f_A(x) \geqslant f_B(x)$;
(c) *intersection:* $C = A \wedge B$ iff $\forall x \in U, t_C(x) = t_A(x) \wedge t_B(x) = inf(t_A(x), t_B(x))$ and $f_C(x) = f_A(x) \vee f_B(x) = sup(f_A(x), f_B(x))$;
(d) *union:* $D = A \vee B$ iff $\forall x \in U, t_D(x) = t_A(x) \vee t_B(x) = sup(t_A(x), t_B(x))$ and $f_D(x) = f_A(x) \wedge f_B(x) = inf(f_A(x), f_B(x))$;
(e) *complement:* $A^c = (A)^c$ iff $\forall x \in U, t_{A^c}(x) = f_A(x)$ and $f_{A^c}(x) = t_A(x)$.

2.2. Fuzzy relation and fuzzy approximation space

Let U, V be two non-empty and finite universes, $\mathcal{F}(U)$ denotes all the fuzzy sets on $U, \mathcal{F}(U \times V)$ denotes all the fuzzy relations on $U \times V$. Especially, if $U = V, \mathcal{F}(U \times U)$ be called the binary fuzzy relation of U.

Usually, for the finite universes $U = \{u_1, u_2, \ldots, u_m\},\ V = \{v_1, v_2, \ldots, v_n\}$, the fuzzy relation $R \in \mathcal{F}(U \times V)$ may be expressed by the fuzzy matrix, i.e., if the element $r_{ij} = R(u_i, v_j)$, then the matrix $R = (r_{ij})_{m \times n}$ represents a fuzzy relation on $U \times V$. In general, $0 \leqslant r_{ij} \leqslant 1$, if $r_{ij} = 0, 1$, the fuzzy matrix R will becomes a Boolean matrix.

Definition 2.2. Suppose $R = (r_{ij})_{m \times n}$ is a fuzzy matrix, for all $\lambda \in [0,1]$, the λ-cut matrix of R is defined as follows:

$$R_\lambda = (r_{ij}(\lambda))_{m \times n},$$

where

$$r_{ij}(\lambda) = \begin{cases} 1, r_{ij} \geqslant \lambda, \\ 0, r_{ij} < \lambda. \end{cases}$$

One can see from the above definition that the λ-cut matrix R_λ indicates the λ-cut relation, i.e., for all $(u, v) \in U \times V$

$$R_\lambda(u, v) = 1 \iff R(u, v) \geqslant \lambda, \quad \forall \lambda \in [0, 1].$$

Obviously, the λ-cut matrix is a Boolean matrix.

Definition 2.3. Suppose $R \in \mathcal{F}(U \times U)$, if the following conditions are satisfied, namely,

(a1) *Reflexivity:* $R(u, u) = 1, \ \forall u \in U$;
(b1) *Symmetry:* $R(u, v) = R(v, u), \ \forall u, v \in U$;
(c1) *Transitivity:* $R(u, v) \geqslant \bigvee_{w \in U} (R(u, w) \wedge R(w, v)), \ \forall u, v \in U$;

then the relation R be called the *fuzzy equivalence relation* on U. The pair (U, R) is called the *fuzzy approximation space*.

In addition, if $R \in \mathcal{F}(U \times U)$ and R satisfies reflexivity and symmetry, then R is called the *fuzzy similarity relation*.

Generally, if the universe U is a finite set, then the fuzzy equivalence relation R can be expressed by a fuzzy equivalence matrix. By Definition 2.3, if $R = (r_{ij})_{n \times n}$ is a fuzzy matrix, then the satisfied conditions will change into the following expressions:

(a2) *Reflexivity:* $r_{ii} = 1$;
(b2) *Symmetry:* $r_{ij} = r_{ji}$;
(c2) *Transitivity:* $r_{ij} \geqslant \bigvee_{k=1}^{n} (r_{ik} \wedge r_{kj})$.

If a Boolean matrix R has the reflexivity, symmetry and transitivity, then R is called an equivalent Boolean matrix, and it expresses an ordinary equivalence relation.

Additionally, let R be a fuzzy equivalence relation, for all $\lambda \in [0, 1]$, we know that the λ-cut matrix R_λ is an equivalent Boolean matrix. Moreover, we have the following conclusion.

Lemma 2.1. *Let $R \in \mathcal{F}(U \times U)$ be a fuzzy relation, R is a fuzzy equivalence relation if and only if R_λ is a fuzzy equivalence relation for all $\lambda \in [0, 1]$.*

If R is a fuzzy equivalence relation, then the λ-cut relation R_λ satisfies the following property.

Lemma 2.2. *Let $R \in \mathcal{F}(U \times U)$ be a fuzzy equivalence relation, for all $\lambda, \mu \in [0, 1]$, if $\lambda < \mu$, then $R_\lambda \supseteq R_\mu$.*

Furthermore, we briefly introduce several properties of operations with respect to two fuzzy equivalence relations.

Lemma 2.3. *Let $R, S \in \mathcal{F}(U \times U)$ be two fuzzy equivalence relations, the following conclusions hold.*

(i) $R \cap S$ is still a fuzzy equivalence relation;
(ii) $R \cup S$ is a fuzzy similarity relation; (not a fuzzy equivalence relation)
(iii) If $R \cup S$ satisfies the transitivity, then $R \cup S$ is a fuzzy equivalence relation.

2.3. β-Complement operator and β-complement product

In this subsection, we recall the definition of the β-operator and introduce a new concept – β-complement operator at first. Afterwards, several existed properties of β-operator are reviewed. And some new properties are examined. Furthermore, we define the β-product and β-complement product between two fuzzy matrixes in order to calculate conveniently.

Definition 2.4. Suppose $\beta : [0, 1] \times [0, 1] \to [0, 1]$, for all $a, b \in [0, 1]$, we have

$$a\beta b = \begin{cases} \frac{b}{a}, & a > b, \\ 1, & a \leqslant b, \end{cases} \tag{1}$$

the mapping β is called the *β-operator.*

Now, let us introduce a new operator given by

$$a\beta^* b = \begin{cases} 1 - \frac{b}{a}, & a > b, \\ 0, & a \leqslant b, \end{cases} \tag{2}$$

where $a, b \in [0, 1]$.

According to Definition 2.4, it stands to reason that $a\beta b + a\beta^* b = 1$ for all $a, b \in [0, 1]$. Therefore, the operator β^* is called the *complement operator* of β. For short, it can be called β-*complement operator*.

Next, we will recall and educe some new properties of the β-operator, which will provide a foundation for the proofs of the properties of approximation operators in Section 3.

Lemma 2.4. *For all $a, b, c \in [0, 1]$, the following conclusions hold.*

(L1) If $b \leqslant c$, then $a\beta b \leqslant a\beta c$;
(L2) $(a\beta b) \wedge (a\beta c) = a\beta(b \wedge c)$;
(L3) $(a \vee c)\beta a = ((a \wedge b) \vee c)\beta a$.

Theorem 2.1. *For all $a, b, c \in [0, 1]$, the inequality $(a \vee c)\beta a \leqslant (a \vee b \vee c)\beta(a \vee b)$ holds.*

Proof. The proof is divided into two cases.

(i) When $c \geqslant (a \vee b),\ a \vee c = c$ and $a \vee b \vee c = c$, by Definition 2.4, we have

$$(a \vee b \vee c)\beta(a \vee b) = c\beta(a \vee b) \geqslant c\beta a = (a \vee c)\beta a.$$

(ii) When $c < (a \vee b),\ a \vee b \vee c = a \vee b$, we can obtain

$$(a \vee b \vee c)\beta(a \vee b) = (a \vee b)\beta(a \vee b) = 1 \geqslant (a \vee c)\beta a.$$

By summing up the previous two parts, we have thus proved the theorem. □

Theorem 2.2. *For all $a, b \in [0, 1]$, the equality $a\beta b = (a \vee b)\beta b$ holds.*

Proof. The proof will be divided into two parts.

(i) When $a \leqslant b,\ a \vee b = b$, we have $(a \vee b)\beta b = b\beta b = 1 = a\beta b$.
(ii) When $a > b,\ a \vee b = a$, we have $(a \vee b)\beta b = a\beta b$.

It is now obvious that the theorem holds. □

Theorem 2.3. *For all $a, b, c \in [0, 1]$, the equality $(a\beta b) \vee (a\beta c) = a\beta(b \vee c)$ holds.*

Proof. The proof will be partitioned into four cases.

(i) When $a \leqslant b,\ a \leqslant c$, we know that $a \leqslant (b \vee c)$. Hence, we can obtain

$$a\beta b = 1, \quad a\beta c = 1 \quad \text{and} \quad a\beta(b \vee c) = 1.$$

It is easy to see that $(a\beta b) \vee (a\beta c) = a\beta(b \vee c)$.
(ii) When $b < a \leqslant c$, we know that $b \vee c = c$. Therefore, we have

$$(a\beta b) \vee (a\beta c) = \frac{b}{a} \vee 1 = 1, \quad a\beta(b \vee c) = a\beta c = 1.$$

Hence, we can easily know that $(a\beta b) \vee (a\beta c) = a\beta(b \vee c)$.
(iii) When $a > b \geqslant c$, we know that $b \vee c = b$. Hence, we have

$$(a\beta b \vee (a\beta c)) = \frac{b}{a} \vee \frac{c}{a} = \frac{b}{a} = a\beta b = a\beta(b \vee c).$$

(iv) When $a > c \geqslant b$, we can obtain $b \vee c = c$. Thus, we have

$$(a\beta b \vee (a\beta c)) = \frac{b}{a} \vee \frac{c}{a} = \frac{c}{a} = a\beta c = a\beta(b \vee c).$$

We summarize the previous four parts, it can easily be seen that the theorem holds. □

For the need of the following narration, we introduce the β-product and β-complement product between two fuzzy matrixes using the Definition 2.4.

Definition 2.5. Let $A=(a_{ij})_{m\times n},\ B=(b_{ij})_{n\times s}$ be two fuzzy matrixes, the *β-product* of the matrixes A and B is defined as

$$A\beta B = C = (c_{ij})_{m\times s},$$

where $c_{ij}=\bigwedge\limits_{k=1}^{n}\left((a_{ik}\vee b_{kj})\beta b_{kj}\right),\ i=1,2,\ldots,m;\ j=1,2,\ldots,s.$

Definition 2.6. Let $A=(a_{ij})_{m\times n},\ B=(b_{ij})_{n\times s}$ be two fuzzy matrixes, the *β-complement product* of the matrixes A and B is defined as

$$A\beta^{*}B = D = (d_{ij})_{m\times s},$$

where $d_{ij}=\bigvee\limits_{k=1}^{n}\left(a_{ik}\beta^{*}(1-b_{kj})\right),\ i=1,2,\ldots,m;\ j=1,2,\ldots,s.$

By the above definitions, one can see that the conditions of the β-product and β-complement product are the number of columns in the left fuzzy matrix is equal to the number of rows in the right fuzzy matrix. In fact, this case is similar to the multiplication of the ordinary matrix.

Next, we cite an example to illustrate the above definitions. If two fuzzy matrixes A and B are given as follows:

$$A=\begin{bmatrix}0.3 & 0.5\\ 0.2 & 0.7\\ 0.5 & 0.4\end{bmatrix},\quad B=\begin{bmatrix}0.5 & 0.6 & 0.4\\ 0.1 & 0.7 & 0.2\end{bmatrix}.$$

Therefore, we can obtain the following two β-products

$$A\beta B=\begin{bmatrix}\frac{1}{5} & 1 & \frac{2}{5}\\ \frac{1}{7} & 1 & \frac{2}{7}\\ \frac{1}{4} & 1 & \frac{1}{2}\end{bmatrix},\quad B\beta A=\begin{bmatrix}\frac{1}{3} & 1\\ \frac{2}{7} & 1\end{bmatrix}.$$

Meantime, we have the following two β-complement products

$$A\beta^{*}B=\begin{bmatrix}0 & \frac{2}{5} & 0\\ 0 & \frac{4}{7} & 0\\ 0 & \frac{1}{4} & 0\end{bmatrix},\quad B\beta^{*}A=\begin{bmatrix}0 & \frac{1}{2}\\ 0 & \frac{4}{7}\end{bmatrix}.$$

The results show that the β-product does not satisfy the commutative law. In general, even if $A\beta B$ exists, $B\beta A$ does not necessarily exist. For the β-complement product, we have the same conclusions.

3. Rough approximations of vague sets and its properties

In this section, we will define the rough approximations of a vague set in fuzzy approximation space and discuss some properties of the approximation operators. In essence, the problem was already proposed by Wang et al. [29]. Here, we will try to give some answers.

3.1. Lower and upper approximations of vague sets in fuzzy approximation space

Now, let us consider a vague set in fuzzy approximation space and come to the definition of the lower and upper approximations of a vague set based on β-operator and β-complement operator.

Definition 3.1. Let $A=\{\langle x,t_A(x),f_A(x)\rangle : x\in U\}$ be a vague set of the universe of discourse U and let R be a fuzzy equivalence relation on U. The lower approximation $\underline{A}$ and the upper approximation $\overline{A}$ of A in the fuzzy approximation space (U,R) are defined, respectively, by

$$\underline{A}=\{\langle x,t_{\underline{A}}(x),f_{\underline{A}}(x)\rangle : x\in U\},\tag{3}$$

$$\overline{A}=\{\langle x,t_{\overline{A}}(x),f_{\overline{A}}(x)\rangle : x\in U\},\tag{4}$$

where $\forall x \in U$

$$t_{\underline{A}}(x) = \bigwedge_{y\in U} ((R(x,y) \vee t_A(y))\beta t_A(y)),$$

$$f_{\underline{A}}(x) = \bigvee_{y\in U} (R(x,y)\beta^*(1 - f_A(y))),$$

$$t_{\overline{A}}(x) = \bigvee_{y\in U} (R(x,y)\beta^*(1 - t_A(y))),$$

$$f_{\overline{A}}(x) = \bigwedge_{y\in U} ((R(x,y) \vee f_A(y))\beta f_A(y)).$$

In fact, one can see that the lower approximation $\underline{A}$ and the upper approximation $\overline{A}$ are still vague sets. This assertion will be proved by the following theorems.

Theorem 3.1. *Let A be a vague set of the universe of discourse U and let R be a fuzzy equivalence relation on U. The symbols $\underline{A}$ and $\overline{A}$ denote lower approximation and upper approximation of A defined by Definition 3.1, respectively. For all $x \in U$, the truth-membership function and false-membership function of the two approximation operators satisfy the following relationships*

$$t_{\underline{A}}(x) + f_{\underline{A}}(x) \leqslant 1, \tag{5}$$
$$t_{\overline{A}}(x) + f_{\overline{A}}(x) \leqslant 1. \tag{6}$$

Proof. According to Definition 3.1, for all $x \in U$, we have

$$\begin{aligned} 1 - f_{\underline{A}}(x) &= 1 - \bigvee_{y\in U} (R(x,y)\beta^*(1 - f_A(y))) = \bigwedge_{y\in U} [1 - (R(x,y)\beta^*(1 - f_A(y)))] = \bigwedge_{y\in U} (R(x,y)\beta(1 - f_A(y))) \\ &\geqslant \bigwedge_{y\in U} (R(x,y)\beta t_A(y)) \ (\text{Lemma 2.4 (L1), since } 1 - f_A(y) \geqslant t_A(y)) \\ &= \bigwedge_{y\in U} ((R(x,y) \vee t_A(y))\beta t_A(y)) \ (\text{Theorem 2.2}) = t_{\underline{A}}(x). \end{aligned}$$

Hence, it is easy to verify that $t_{\underline{A}}(x) + f_{\underline{A}}(x) \leqslant 1$ for all $x \in U$. Evidently, the other conclusion can be proved in a similar way as shown before. □

Notice that the Theorem 3.1 shows that the expressions of truth-membership function and false-membership function in Definition 3.1 are reasonable. Besides, it can be easily verified that the expressions $t_{\underline{A}}(x)$ and $f_{\overline{A}}(x)$ can be abbreviated, by Lemma 2.4, as

$$t_{\underline{A}}(x) = \bigwedge_{y\in U} (R(x,y)\beta t_A(y)),$$

$$f_{\overline{A}}(x) = \bigwedge_{y\in U} (R(x,y)\beta f_A(y)).$$

Next, we will discuss two special cases of Definition 3.1. First of all, if $A \in \mathcal{F}(U)$ is a fuzzy set, then it can be written as the following form, i.e., $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, and satisfying $\mu_A(x) + \nu_A(x) = 1$ for all $x \in U$. Through the above definition, the truth-membership functions of the lower approximation $\underline{A}$ and the upper approximations $\overline{A}$ of A can be obtained as follows:

$$\mu_{\underline{A}}(x) = \bigwedge_{y\in U} \big((R(x,y) \vee \mu_A(y))\beta\mu_A(y)\big) = \bigwedge_{y\in U} \big(R(x,y)\beta\mu_A(y)\big), \quad \forall x \in U,$$

$$\mu_{\overline{A}}(x) = \bigvee_{y\in U} \big(R(x,y)\beta^*(1 - \mu_A(y))\big), \forall x \in U.$$

Therefore, for all $x \in U$, we have

$$1 - \mu_{\underline{A}}(x) = 1 - \bigwedge_{y\in U} (R(x,y)\beta\mu_A(y)) = \bigvee_{y\in U} (1 - R(x,y)\beta\mu_A(y)) = \bigvee_{y\in U} \big(R(x,y)\beta^*\mu_A(y)\big) = \bigvee_{y\in U} (R(x,y)\beta^*(1 - \nu_A(y))) = \nu_{\underline{A}}(x).$$

Analogously, we can obtain $\mu_{\overline{A}}(x) + \nu_{\overline{A}}(x) = 1$ for all $x \in U$. In essence, one can see from the previous proof of Theorem 3.1 that these conclusions hold.

From the above statement, it is obvious that the lower approximation and upper approximation of a fuzzy set are still fuzzy sets in fuzzy environment. Generally, the pair $(\underline{A}, \overline{A})$ can be called fuzzy rough sets in the fuzzy approximation space. And it should be pointed out that the expressions of the lower and upper approximations are different from the ones proposed by Liu [31,32].

Furthermore, when R is an equivalence relation on U, i.e., the corresponding relation matrix R is an equivalent Boolean matrix, for all $x, y \in U$, we have $R(x,y) = 1$ or $R(x,y) = 0$. Based on the relationship between the equivalence relation and the equivalence class, we draw a conclusion

$$R(x,y) = \begin{cases} 1, & y \in [x]_R, \\ 0, & y \notin [x]_R, \end{cases}$$

where $[x]_R$ denotes the equivalence class which belongs to x.

Consequently, by Definition 3.1, for a vague set A, the truth-membership function and false-membership function of the lower approximation $\underline{A}$ and the upper approximation $\overline{A}$ can be further simplified. For all $x \in U$

$$\begin{aligned} t_{\underline{A}}(x) &= \bigwedge_{y\in U} ((R(x,y) \vee t_A(y))\beta t_A(y)) = \left(\bigwedge_{y\in[x]_R} ((R(x,y) \vee t_A(y))\beta t_A(y))\right) \bigwedge \left(\bigwedge_{y\notin[x]_R} ((R(x,y) \vee t_A(y))\beta t_A(y))\right) \\ &= \left(\bigwedge_{y\in[x]_R} (1\beta t_A(y))\right) \bigwedge \left(\bigwedge_{y\notin[x]_R} (t_A(y)\beta t_A(y))\right) = \left(\bigwedge_{y\in[x]_R} (t_A(y))\right) \bigwedge \left(\bigwedge_{y\notin[x]_R} 1\right) = \bigwedge_{y\in[x]_R} t_A(y) = inf\{t_A(y) : y \in [x]_R\}, \end{aligned}$$

$$\begin{aligned} f_{\underline{A}}(x) &= \bigvee_{y\in U} (R(x,y)\beta^*(1 - f_A(y))) = \left(\bigvee_{y\in[x]_R} (R(x,y)\beta^*(1 - f_A(y)))\right) \bigvee \left(\bigvee_{y\notin[x]_R} (R(x,y)\beta^*(1 - f_A(y)))\right) \\ &= \left(\bigvee_{y\in[x]_R} (1\beta^*(1 - f_A(y)))\right) = \bigvee_{y\in[x]_R} f_A(y) = sup\{f_A(y) : y \in [x]_R\}. \end{aligned}$$

Similarly, we can obtain the following two expressions by the previous inferences

$$\begin{aligned} t_{\overline{A}}(x) &= sup\{t_A(y) : y \in [x]_R\}, \\ f_{\overline{A}}(x) &= inf\{f_A(y) : y \in [x]_R\}. \end{aligned}$$

From the previous results, we can see that the lower approximation $\underline{A}$ and the upper approximation $\overline{A}$ turn into the rough approximations, which defined by Wang and Al-Rababah [29,30], in Pawlak approximation space.

By summarizing the above analysis, it is not difficult to find out that the Definition 3.1 is not only an extension of the concept of the fuzzy rough sets, but also can be viewed as a generalization of the rough vague sets in Pawlak approximation space.

This case is the same as the rough vague sets in Pawlak approximation space, and then we can introduce the concept of the rough vague sets in fuzzy approximation space by Definition 3.1.

Definition 3.2. Let A be a vague set of the universe of discourse U and let R be a fuzzy equivalence relation on U. The pair (U, R) is a fuzzy approximation space, $\underline{A}$ and $\overline{A}$ denote, respectively, the lower approximation and the upper approximation of A. If $\underline{A} = \overline{A}$, we call the vague set A is a *definable vague set* in (U, R). Otherwise, it is to be called a *rough vague set* in (U, R).

In order not to be confused with the rough vague sets in Pawlak approximation space, here, the rough vague sets are called *fuzzy rough vague sets* in fuzzy approximation space. Universally, the pair $(\underline{A}, \overline{A})$ is called fuzzy rough vague set, which is FRVS in abbreviation.

Additionally, if R is a fuzzy equivalence relation on U, according to Definition 2.2, for all $\lambda \in [0,1]$, the λ-cut relation R_λ is an equivalence relation on U. Therefore, we can define the rough approximations of a vague set using the equivalence relation R_λ.

The following definition can be obtained with the same principle as rough vague sets in Pawlak approximation.

Definition 3.3. Let $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in U\}$ be a vague set of the universe of discourse U and let R be a fuzzy equivalence relation on U. For all $\lambda \in [0,1]$, the λ-lower approximation $\underline{A}_{R_\lambda}$ and the λ-upper approximation $\overline{A}_{R_\lambda}$ of A in the fuzzy approximation space (U, R) are defined, respectively, by

$$\underline{A}_{R_\lambda} = \left\{\left\langle x, t_{\underline{A}_{R_\lambda}}(x), f_{\underline{A}_{R_\lambda}}(x)\right\rangle : x \in U\right\}, \tag{7}$$

$$\overline{A}_{R_\lambda} = \left\{\left\langle x, t_{\overline{A}_{R_\lambda}}(x), f_{\overline{A}_{R_\lambda}}(x)\right\rangle : x \in U\right\}, \tag{8}$$

where $\forall x \in U$

$$\begin{aligned} t_{\underline{A}_{R_\lambda}}(x) &= inf\{t_A(y) : y \in [x]_{R_\lambda}\}, \\ f_{\underline{A}_{R_\lambda}}(x) &= sup\{f_A(y) : y \in [x]_{R_\lambda}\}, \\ t_{\overline{A}_{R_\lambda}}(x) &= sup\{t_A(y) : y \in [x]_{R_\lambda}\}, \\ f_{\overline{A}_{R_\lambda}}(x) &= inf\{f_A(y) : y \in [x]_{R_\lambda}\}. \end{aligned}$$

Here, $[x]_{R_\lambda}$ is the equivalence class of the element x under the equivalence relation R_λ.

3.2. The properties of lower and upper approximations of vague sets

In this subsection, we mainly study the properties of the lower and upper approximations of the vague sets in fuzzy approximation space. Some properties of λ-lower and λ-upper approximations of the vague sets are discussed, which have not been discussed in [29,30].

Theorem 3.2. *Let A, B be two vague sets of U, then the following properties are satisfied by the lower and upper approximations of A, B:*

(P1) $\underline{A} \subseteq A \subseteq \overline{A}$;
(P2) If $A \subseteq B$, then $\underline{A} \subseteq \underline{B}$, $\overline{A} \subseteq \overline{B}$;
(P3) $\overline{A \vee B} = \overline{A} \vee \overline{B}$, $\underline{A \wedge B} = \underline{A} \wedge \underline{B}$;
(P4) $\overline{A \wedge B} \subseteq \overline{A} \wedge \overline{B}$, $\underline{A \vee B} \supseteq \underline{A} \vee \underline{B}$;
(P5) $\underline{A^c} = \overline{A}^c$, $\overline{A^c} = \underline{A}^c$.

Proof.

(P1) For all $x \in U$, by Definition 3.1, we know that

$$t_{\underline{A}}(x) = \bigwedge_{y\in U}((R(x,y) \vee t_A(y))\beta t_A(y)) = \bigwedge_{\substack{y\in U\\ y\neq x}}((R(x,y) \vee t_A(y))\beta t_A(y)) \bigwedge ((R(x,x) \vee t_A(x))\beta t_A(x))$$
$$= \bigwedge_{\substack{y\in U\\ y\neq x}}((R(x,y) \vee t_A(y))\beta t_A(y)) \bigwedge t_A(x) \leqslant t_A(x),$$

$$f_{\underline{A}}(x) = \bigvee_{y\in U}(R(x,y)\beta^*(1-f_A(y))) = \bigvee_{\substack{y\in U\\ y\neq x}}(R(x,y)\beta^*(1-f_A(y))) \bigvee (R(x,x)\beta^*(1-f_A(x))) = \bigvee_{\substack{y\in U\\ y\neq x}}(R(x,y)\beta^*(1-f_A(y))) \bigvee f_A(x)$$
$$\geqslant f_A(x).$$

Similarly, we can also obtain $t_{\overline{A}}(x) \geqslant t_A(x)$ and $f_{\overline{A}}(x) \leqslant f_A(x)$, namely, we have $\underline{A} \subseteq A \subseteq \overline{A}$.

(P2) Since $A \subseteq B$, it satisfies that $t_A(x) \leqslant t_B(x)$ and $f_A(x) \geqslant f_B(x)$ for all $x \in U$. Now, we start to prove the conclusion.

Case 1 If there exists $x \in U$ such that $t_A(y) \geqslant R(x,y)$ for every $y \in U$, then we can easily get $t_B(y) \geqslant t_A(y) \geqslant R(x,y)$. Since R is an fuzzy equivalence relation, it follows $R(x,x) = 1$. Therefore, we can know that $t_A(x) = t_B(x) = 1$. Hence, it is easy to show that $t_{\underline{A}}(x) = t_{\underline{B}}(x) = 1$.

Case 2 For every $x \in U$, there exists $y \in U$ such that $t_A(y) < R(x,y)$.

(i) If $R(x,y) \leqslant t_B(y)$ for each y such that $t_A(y) < R(x,y)$, we can obtain $t_B(x) = 1$ for all $x \in U$. Thus, we have $t_{\underline{A}}(x) \leqslant t_{\underline{B}}(x) = 1$.

(ii) If $t_A(y) \leqslant t_B(y) < R(x,y)$, then we have

$$t_{\underline{A}}(x) = \bigwedge_{y\in U}((R(x,y) \vee t_A(y))\beta t_A(y))$$
$$= \left(\bigwedge_{t_A(y)<R(x,y)}((R(x,y) \vee t_A(y))\beta t_A(y))\right) \bigwedge \left(\bigwedge_{t_A(y)\geqslant R(x,y)}((R(x,y) \vee t_A(y))\beta t_A(y))\right) \bigwedge_{t_A(y)<R(x,y)}((R(x,y) \vee t_A(y))\beta t_A(y))$$
$$= \bigwedge_{t_A(y)<R(x,y)} \frac{t_A(y)}{R(x,y)} \leqslant \bigwedge_{t_B(y)<R(x,y)} \frac{t_A(y)}{R(x,y)} (t_A(x) \leqslant t_B(x)) \leqslant \bigwedge_{t_B(y)<R(x,y)} \frac{t_B(y)}{R(x,y)} = t_{\underline{B}}(x) \quad \text{(Lemma 2.4 (L1))}.$$

By the above two cases, it is now obvious that $t_{\underline{A}}(x) \leqslant t_{\underline{B}}(x)$ for all $x \in U$.
Moreover, for all $x \in U$, since $f_A(x) \geqslant f_B(x) \Rightarrow 1 - f_A(x) \leqslant 1 - f_B(x)$. According to Lemma 2.4 (L1), we have

$$R(x,y)\beta(1-f_A(y)) \leqslant R(x,y)\beta(1-f_B(y)) \quad \text{for all } x,y \in U.$$

We then can obtain

$$f_{\underline{A}}(x) = \bigvee_{y\in U}(R(x,y)\beta^*(1-f_A(y))) = \bigvee_{y\in U}[1-(R(x,y)\beta(1-f_A(y)))] \geqslant \bigvee_{y\in U}[1-(R(x,y)\beta(1-f_B(y)))]$$
$$= \bigvee_{y\in U}(R(x,y)\beta^*(1-f_B(y))) = f_{\underline{B}}(x).$$

Hence, $\underline{A} \subseteq \underline{B}$. Similarly, we can prove that $\overline{A} \subseteq \overline{B}$ in the same way as shown before.

(P3) For all $x \in U$

$$\begin{aligned} t_{\overline{A \vee B}}(x) &= \bigvee_{y \in U} (R(x,y)\beta^*(1 - t_{A\vee B}(y))) = \bigvee_{y \in U}(R(x,y)\beta^*(1 - (t_A(y) \vee t_B(y)))) = \bigvee_{y \in U} (R(x,y)\beta^*((1 - t_A(y)) \wedge (1 - t_B(y)))) \\ &= \bigvee_{y \in U} [1 - (R(x,y)\beta((1 - t_A(y)) \wedge (1 - t_B(y))))] \text{ (Lemma 2.4 (L2))} \\ &= \bigvee_{y \in U} [1 - (R(x,y) \wedge (1 - t_A(y))) \wedge (R(x,y) \wedge (1 - t_B(y)))] \\ &= \bigvee_{y \in U} [(1 - R(x,y)\beta(1 - t_A(y))) \vee (1 - R(x,y)\beta(1 - t_B(y)))] \\ &= \left(\bigvee_{y \in U}(1 - R(x,y)\beta(1 - t_A(y)))\right) \vee \left(\bigvee_{y \in U}(1 - R(x,y)\beta(1 - t_A(y)))\right) \\ &= \left(\bigvee_{y \in U}(R(x,y)\beta^*(1 - t_A(y)))\right) \vee \left(\bigvee_{y \in U}(R(x,y)\beta^*(1 - t_B(y)))\right) = t_{\overline{A}}(x) \vee t_{\overline{B}}(x) = t_{\overline{A}\vee\overline{B}}(x), \end{aligned}$$

$$\begin{aligned} f_{\overline{A}\vee\overline{B}}(x) &= f_{\overline{A}}(x) \wedge f_{\overline{B}}(x) = \left(\bigwedge_{y \in U}((R(x,y) \vee f_A(y))\beta f_A(y))\right) \wedge \left(\bigwedge_{y \in U}((R(x,y) \vee f_B(y))\beta f_B(y))\right) \\ &= \left(\bigwedge_{y \in U} ((R(x,y) \vee (f_A(y) \wedge f_B(y)))\beta f_A(y))\right) \wedge \left(\bigwedge_{y \in U} ((R(x,y) \vee (f_A(y) \wedge f_B(y)))\beta f_B(y))\right) \text{ (Lemma 2.4 (L3))} \\ &= \bigwedge_{y \in U} ((R(x,y) \vee (f_A(y) \wedge f_B(y)))\beta(f_A(y) \wedge f_B(y))) \text{ (Lemma 2.4 (L2))} = \bigwedge_{y \in U} ((R(x,y) \vee f_{A\vee B}(y))\beta f_{A\vee B}(y)) = f_{\overline{A\vee B}}(x). \end{aligned}$$

Hence, $\overline{A \vee B} = \overline{A} \vee \overline{B}$. Similarly, we can obtain $\underline{A \wedge B} = \underline{A} \wedge \underline{B}$ by the same way.

(P4) Here, we give two methods to prove this conclusion.

(a) We can see from Definition 2.1 that $A \wedge B \subseteq A$, $A \wedge B \subseteq B$. By the previous property (P2), we can obtain $\overline{A \wedge B} \subseteq \overline{A}$ and $\overline{A \wedge B} \subseteq \overline{B}$. Therefore, $\overline{A \wedge B} \subseteq \overline{A} \wedge \overline{B}$. Similarly, $\underline{A \vee B} \supseteq \underline{A} \vee \underline{B}$ can also be proved.

(b) For all $x \in U$

$$\begin{aligned} t_{\overline{A}\wedge\overline{B}}(x) &= t_{\overline{A}}(x) \wedge t_{\overline{B}}(x) = \left(\bigvee_{y \in U} (R(x,y)\beta^*(1 - t_A(y)))\right) \wedge \left(\bigvee_{y \in U} (R(x,y)\beta^*(1 - t_B(y)))\right) \\ &\geqslant \bigvee_{y \in U} ((R(x,y)\beta^*(1 - t_A(y))) \wedge (R(x,y)\beta^*(1 - t_B(y)))) = \bigvee_{y \in U} ((1 - R(x,y)\beta(1 - t_A(y))) \wedge (1 - R(x,y)\beta(1 - t_B(y)))) \\ &= \bigvee_{y \in U} [1 - ((R(x,y)\beta(1 - t_A(y))) \vee (R(x,y)\beta(1 - t_B(y))))] \\ &= \bigvee_{y \in U} [1 - R(x,y)\beta((1 - t_A(y)) \vee (1 - t_B(y)))] \text{ (Theorem 2.3)} = \bigvee_{y \in U} [1 - R(x,y)\beta(1 - (t_A(y) \wedge t_B(y)))] \\ &= \bigvee_{y \in U} [1 - R(x,y)\beta(1 - t_{A\wedge B}(y))] = \bigvee_{y \in U} (R(x,y)\beta^*(1 - t_{A\wedge B}(y))) = t_{\overline{A\wedge B}}(x), \end{aligned}$$

$$\begin{aligned} f_{\overline{A}\wedge\overline{B}}(x) &= f_{\overline{A}}(x) \vee f_{\overline{B}}(x) = \left(\bigwedge_{y \in U}((R(x,y) \vee f_A(y))\beta f_A(y))\right) \vee \left(\bigwedge_{y \in U}((R(x,y) \vee f_B(y))\beta f_B(y))\right) \\ &\leqslant \bigwedge_{y \in U} (((R(x,y) \vee f_A(y))\beta f_A(y)) \vee ((R(x,y) \vee f_B(y))\beta f_B(y))) \\ &\leqslant \bigwedge_{y \in U} (((R(x,y) \vee (f_A(y) \vee f_B(y)))\beta(f_A(y) \vee f_B(y))) \vee ((R(x,y) \vee (f_B(y) \vee f_A(y)))\beta(f_B(y) \vee f_A(y)))) \text{ (Theorem 2.1)} \\ &= \bigwedge_{y \in U} ((R(x,y) \vee (f_A(y) \vee f_B(y)))\beta(f_A(y) \vee f_B(y))) = \bigwedge_{y \in U} ((R(x,y) \vee f_{A\wedge B}(y))\beta f_{A\wedge B}(y)) = f_{\overline{A\wedge B}}(x). \end{aligned}$$

Therefore, we have $\overline{A \wedge B} \subseteq \overline{A} \wedge \overline{B}$. Similarly, we can obtain $\underline{A \vee B} \supseteq \underline{A} \vee \underline{B}$ by the method analogous to that used above.

(P5) Since $t_{A^c}(x) = f_A(x)$, $f_{A^c}(x) = t_A(x)$, for all $x \in U$, we know that

$$\begin{aligned} t_{\underline{A^c}}(x) &= \bigwedge_{y \in U} (R(x,y) \vee t_{A^c}(y))\beta t_{A^c}(y) = \bigwedge_{y \in U}(R(x,y) \vee f_A(y))\beta f_A(y) = f_{\overline{A}}(x) = t_{\overline{A}^c}(x), \\ f_{\underline{A^c}}(x) &= \bigvee_{y \in U} (R(x,y)\beta^*(1 - f_{A^c}(y))) = \bigvee_{y \in U} (R(x,y)\beta^*(1 - t_A(y))) = t_{\overline{A}}(x) = f_{\overline{A}^c}(x). \end{aligned}$$

Hence, we have $\underline{A^c} = \overline{A}^c$. In addition, we can also prove that $\overline{A^c} = \underline{A}^c$ by the same way. □

From the above conclusions, it can be observed that the lower and upper approximations of vague sets in fuzzy approximation space have the same properties as the lower and upper approximations of the classical rough sets, fuzzy rough sets, rough fuzzy sets and vague sets defined in Pawlak approximation space.

Theorem 3.3. *Let A be a vague set of U and let R be a fuzzy equivalence relation on U, then the lower and upper approximations satisfy $\underline{(\underline{A})} = \underline{A}$ and $\overline{(\overline{A})} = \overline{A}$, respectively.*

Proof. By Theorem 3.2 (P1,P2), we can easily know that

$$\underline{(\underline{A})} \subseteq \underline{A} \quad \text{and} \quad \overline{(\overline{A})} \supseteq \overline{A}.$$

Similar to the proof of Theorem 3.2 (P2), we also will divide into two cases to prove another part of the theorem.

First of all, we start to prove the conclusion that $t_{\underline{(\underline{A})}}(x) \geqslant t_{\underline{A}}(x)$ for all $x \in U$.

Case 1 If there exists $x \in U$ such that $t_{\underline{A}}(y) \geqslant R(x,y)$ for every $y \in U$, we then obtain

$$t_A(y) \geqslant t_{\underline{A}}(y) \geqslant R(x,y).$$

Since $R(x,x) = 1$, it is obvious that $t_A(x) = t_{\underline{A}}(x) = 1$.
By Definitions 2.4 and 3.1, it can easily be seen that $t_{\underline{(\underline{A})}}(x) = t_{\underline{A}}(x) = 1$.

Case 2 For every $x \in U$, there exists $y \in U$ such that $t_{\underline{A}}(y) < R(x,y)$.

(i) If $R(x,y) \leqslant t_A(y)$ for each y such that $t_{\underline{A}}(y) < R(x,y)$, we can obtain $t_A(x) = 1$ for all $x \in U$. Therefore, we have $t_{\underline{A}}(x) = 1$ for all $x \in U$.
Thus, for every $y \in U$, it follows $t_{\underline{A}}(x) \geqslant R(x,y)$. Obviously, this leads to a contradiction.

(ii) If $t_{\underline{A}}(y) \leqslant t_A(y) < R(x,y)$, then we have

$$t_{\underline{(\underline{A})}}(x) = \bigwedge_{t_{\underline{A}}(y) < R(x,y)} \frac{t_{\underline{A}}(y)}{R(x,y)}.$$

Therefore, there exists $x_0 \in U$ such that

$$\frac{t_{\underline{A}}(x_0)}{R(x,x_0)} = t_{\underline{(\underline{A})}}(x).$$

Similarly, for $t_{\underline{A}}(x_0) = \bigwedge_{t_A(y) < R(x_0,y)} \frac{t_A(y)}{R(x_0,y)}$, there exists $y_0 \in U$ such that

$$\frac{t_A(y_0)}{R(x_0,y_0)} = t_{\underline{A}}(x_0).$$

Because the fuzzy equivalence relation satisfies transitivity, namely, $R(x,x_0) \wedge R(x_0,y_0) \leqslant R(x,y_0)$, we can obtain

$$t_{\underline{(\underline{A})}}(x) = \frac{t_{\underline{A}}(x_0)}{R(x,x_0)} = \frac{1}{R(x,x_0)} \cdot \frac{t_A(y_0)}{R(x_0,y_0)} \geqslant \frac{t_A(y_0)}{R(x,x_0) \wedge R(x_0,y_0)} \geqslant \frac{t_A(y_0)}{R(x,y_0)} \geqslant t_{\underline{A}}(x).$$

According to the previous proof of two cases, we are led to the conclusion that $t_{\underline{(\underline{A})}}(x) \geqslant t_{\underline{A}}(x)$ for all $x \in U$.

Secondly, we will prove the conclusion that $f_{\underline{(\underline{A})}}(x) \leqslant f_{\underline{A}}(x)$ for all $x \in U$.

Similar to the previous one, the proof can be divided into two cases.

Case 1 For every $y \in U$, if there exists $x \in U$ such that $R(x,y) \leqslant 1 - f_{\underline{A}}(y)$, then we can get $f_{\underline{A}}(x) = 0$.
Since $f_A(x) \leqslant f_{\underline{A}}(x)$, we have $f_A(x) = f_{\underline{A}}(x) = 0$. Therefore, $f_{\underline{(\underline{A})}}(x) = f_{\underline{A}}(x) = 0$.

Case 2 For every $x \in U$, there exists $y \in U$ such that $R(x,y) > 1 - f_{\underline{A}}(y)$.

(i) If $1 - f_A(y) \geqslant R(x,y) > 1 - f_{\underline{A}}(y)$, we then obtain that $f_A(x) = 0$ for all $x \in U$. Therefore, we have $f_{\underline{A}}(x) = 0$ for all $x \in U$.
Hence, for all $x \in U$, $1 - f_{\underline{A}}(x) = 1 \geqslant R(x,y)$. Clearly, this is contrary to the condition.

(ii) If $R(x,y) > 1 - f_A(y) \geqslant 1 - f_{\underline{A}}(y)$, then we have

$$\begin{aligned} f_{\underline{(\underline{A})}}(x) &= \bigvee_{y \in U} \left(R(x,y)\beta^*(1 - f_{\underline{A}}(y))\right) = \bigvee_{y \in U} \left[1 - R(x,y)\beta(1 - f_{\underline{A}}(y))\right] = 1 - \bigwedge_{y \in U} \left(R(x,y)\beta(1 - f_{\underline{A}}(y))\right) \\ &= 1 - \bigwedge_{R(x,y) > 1 - f_{\underline{A}}(y)} \frac{1 - f_{\underline{A}}(y)}{R(x,y)}. \end{aligned}$$

Thus, there exists $x_0 \in U$ such that

$$\frac{1 - f_{\underline{A}}(x_0)}{R(x,x_0)} = 1 - f_{\underline{(\underline{A})}}(x).$$

Certainly, for $1 - f_{\underline{A}}(x_0) = \bigwedge_{R(x_0,y)>1-f_A(y)} \frac{1-f_A(y)}{R(x_0,y)}$, there exists $y_0 \in U$ such that

$$\frac{1 - f_A(y_0)}{R(x_0, y_0)} = 1 - f_{\underline{A}}(x_0).$$

Analogously, we can obtain

$$f_{\underline{(\underline{A})}}(x) = 1 - \frac{1 - f_{\underline{A}}(x_0)}{R(x, x_0)} = 1 - \frac{1}{R(x, x_0)} \cdot \frac{1 - f_A(y_0)}{R(x_0, y_0)} \leqslant 1 - \frac{1 - f_A(y_0)}{R(x, x_0) \wedge R(x_0, y_0)b} \leqslant 1 - \frac{1 - f_A(y_0)}{R(x, y_0)} \text{ (Transitivity)}$$
$$\leqslant 1 + f_{\underline{A}}(x) - 1 = f_{\underline{A}}(x).$$

The proof of the conclusion is now completed. Similarly, we can obtain $\overline{(\overline{A})} = \overline{A}$ by the method analogous to that used above. □

Next, we will discuss some properties of the λ-lower and λ-upper approximations.

Theorem 3.4. *Let A be a vague set of U. R, S are two fuzzy equivalence relations on U. For all $\lambda \in [0, 1]$, if $R \subseteq S$, then the λ-lower and λ-upper approximations satisfy $\underline{A}_{R_\lambda} \supseteq \underline{A}_{S_\lambda}$ and $\overline{A}_{R_\lambda} \subseteq \overline{A}_{S_\lambda}$, respectively.*

Proof. According to Lemmas 2.1 and 2.2, suppose $R = (r_{ij})_{m\times n}$, $S = (s_{ij})_{m\times n}$, if $R \subseteq S$, we have $r_{ij} \leqslant s_{ij}$. Therefore, for all $\lambda \in [0, 1]$, if $r_{ij} \geqslant \lambda$, then $s_{ij} \geqslant \lambda$. So we have $R_\lambda \subseteq S_\lambda$. Meantime, we can obtain $[x]_{R_\lambda} \subseteq [x]_{S_\lambda}$. By Definition 3.3, for all $x \in U$, we have

$$t_{\underline{A}_{R_\lambda}}(x) = \inf\{t_A(x) : y \in [x]_{R_\lambda}\} \geqslant \inf\{t_A(x) : y \in [x]_{S_\lambda}\} = t_{\underline{A}_{S_\lambda}}(x),$$
$$f_{\underline{A}_{R_\lambda}}(x) = \sup\{f_A(x) : y \in [x]_{R_\lambda}\} \leqslant \sup\{f_A(x) : y \in [x]_{S_\lambda}\} = f_{\underline{A}_{S_\lambda}}(x),$$
$$t_{\overline{A}_{R_\lambda}}(x) = \sup\{t_A(x) : y \in [x]_{R_\lambda}\} \leqslant \sup\{t_A(x) : y \in [x]_{S_\lambda}\} = t_{\overline{A}_{S_\lambda}}(x),$$
$$f_{\overline{A}_{R_\lambda}}(x) = \inf\{f_A(x) : y \in [x]_{R_\lambda}\} \geqslant \inf\{f_A(x) : y \in [x]_{S_\lambda}\} = f_{\overline{A}_{S_\lambda}}(x).$$

Hence, $\underline{A}_{R_\lambda} \supseteq \underline{A}_{S_\lambda}$ and $\overline{A}_{R_\lambda} \subseteq \overline{A}_{S_\lambda}$. □

From Lemma 2.3, if R, S are two equivalence relations on U, then the intersection $R \cap S$ must be an equivalence relation on U, but the union $R \cup S$ is not necessary an equivalence relation. The reason is that the transitivity has been untenable. Usually, if the binary relation R satisfies the reflexivity and symmetry, we can construct the transitive closure and make it satisfy the transitivity. As we all know, the transitive closure is a smallest transitive relation containing R, and the transitive closure of R can be denoted by notation R^t. Therefore, the relation $(R \cup S)^t$ is an equivalence relation and the following theorem can be obtained.

Theorem 3.5. *Let A be a vague set of U. R, S are two fuzzy equivalence relations on U. For all $\lambda \in [0, 1]$, the λ-lower and λ-upper approximations satisfy the following expressions.*

(a) $\underline{A}_{(R\cup S)^t_\lambda} = \underline{A}_{(R_\lambda \cup S_\lambda)^t}$ and $\overline{A}_{(R\cup S)^t_\lambda} = \overline{A}_{(R_\lambda \cup S_\lambda)^t}$;
(b) $\underline{A}_{(R\cap S)_\lambda} = \underline{A}_{R_\lambda \cap S_\lambda}$ and $\overline{A}_{(R\cap S)_\lambda} = \overline{A}_{R_\lambda \cap S_\lambda}$;
(c) $\underline{A}_{(R_\lambda \cup S_\lambda)^t} \subseteq \underline{A}_{R_\lambda} \vee \underline{A}_{S_\lambda}$ and $\overline{A}_{(R_\lambda \cup S_\lambda)^t} \supseteq \overline{A}_{R_\lambda} \vee \overline{A}_{S_\lambda}$;
(d) $\underline{A}_{R_\lambda \cap S_\lambda} \supseteq \underline{A}_{R_\lambda} \wedge \underline{A}_{S_\lambda}$ and $\overline{A}_{R_\lambda \cap S_\lambda} \subseteq \overline{A}_{R_\lambda} \wedge \overline{A}_{S_\lambda}$.

Proof. For the conclusions (a) and (b), we need only to prove that $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$ and $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$ for all $\lambda \in [0, 1]$. In fact, suppose $R = (r_{ij})_{m\times n}$, $S = (s_{ij})_{m\times n}$, we have

$$(r_{ij} \vee s_{ij})(\lambda) = 1 \Longleftrightarrow r_{ij} \geqslant \lambda \text{ or } s_{ij} \geqslant \lambda,$$
$$(r_{ij} \wedge s_{ij})(\lambda) = 1 \Longleftrightarrow r_{ij} \geqslant \lambda \text{ and } s_{ij} \geqslant \lambda.$$

Therefore, we know that $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$ and $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$.

Next, we start to prove the conclusions (c) and (d).

(c) For all $x \in U$

$$t_{\underline{A}_{(R_\lambda \cup S_\lambda)^t}}(x) = \inf\left\{t_A(y) : y \in [x]_{(R_\lambda \cup S_\lambda)^t}\right\} \leqslant \inf\{t_A(y) : y \in [x]_{R_\lambda} \cup [x]_{S_\lambda}\} \leqslant \inf\{t_A(y) : y \in [x]_{R_\lambda}\} \vee \inf\{t_A(y) : y \in [x]_{S_\lambda}\}$$
$$= t_{\underline{A}_{R_\lambda}}(x) \vee t_{\underline{A}_{S_\lambda}}(x),$$

$$f_{\underline{A}_{(R_\lambda \cup S_\lambda)^t}}(x) = \sup\left\{f_A(y) : y \in [x]_{(R_\lambda \cup S_\lambda)^t}\right\} \geqslant \sup\{f_A(y) : y \in [x]_{R_\lambda} \cup [x]_{S_\lambda}\} \geqslant \sup\{f_A(y) : y \in [x]_{R_\lambda}\} \wedge \sup\{f_A(y) : y \in [x]_{S_\lambda}\}$$
$$= f_{\underline{A}_{R_\lambda}}(x) \wedge f_{\underline{A}_{S_\lambda}}(x),$$

$$t_{\overline{A}_{(R_\lambda \cup S_\lambda)^t}}(x) = sup\left\{t_A(y) : y \in [x]_{(R_\lambda \cup S_\lambda)^t}\right\} \geqslant sup\{t_A(y) : y \in [x]_{R_\lambda} \cup [x]_{S_\lambda}\} = sup\{t_A(y) : y \in [x]_{R_\lambda}\} \vee sup\{t_A(y) : y \in [x]_{S_\lambda}\}$$
$$= t_{\overline{A}_{R_\lambda}}(x) \vee t_{\overline{A}_{S_\lambda}}(x),$$

$$f_{\overline{A}_{(R_\lambda \cup S_\lambda)^t}}(x) = inf\left\{f_A(y) : y \in [x]_{(R_\lambda \cup S_\lambda)^t}\right\} \leqslant inf\{f_A(y) : y \in [x]_{R_\lambda} \cup [x]_{S_\lambda}\} = inf\{f_A(y) : y \in [x]_{R_\lambda}\} \wedge inf\{f_A(y) : y \in [x]_{S_\lambda}\}$$
$$= f_{\overline{A}_{R_\lambda}}(x) \wedge f_{\overline{A}_{S_\lambda}}(x).$$

According to Definition 2.1, it can be easily shown that the conclusions hold.

(d) For all $x \in U$,

$$t_{\underline{A}_{R_\lambda \cap S_\lambda}}(x) = inf\{t_A(y) : y \in [x]_{R_\lambda \cap S_\lambda}\} = inf\{t_A(y) : y \in [x]_{R_\lambda} \cap [x]_{S_\lambda}\} \geqslant inf\{t_A(y) : y \in [x]_{R_\lambda}\} \wedge inf\{t_A(y) : y \in [x]_{S_\lambda}\}$$
$$= t_{\underline{A}_{R_\lambda}}(x) \wedge t_{\underline{A}_{S_\lambda}}(x),$$

$$f_{\underline{A}_{R_\lambda \cap S_\lambda}}(x) = sup\{f_A(y) : y \in [x]_{R_\lambda \cap S_\lambda}\} = sup\{f_A(y) : y \in [x]_{R_\lambda} \cap [x]_{S_\lambda}\} \leqslant sup\{f_A(y) : y \in [x]_{R_\lambda}\} \vee sup\{f_A(y) : y \in [x]_{S_\lambda}\}$$
$$= f_{\underline{A}_{R_\lambda}}(x) \vee f_{\underline{A}_{S_\lambda}}(x),$$

$$t_{\overline{A}_{R_\lambda \cap S_\lambda}}(x) = sup\{t_A(y) : y \in [x]_{R_\lambda \cap S_\lambda}\} = sup\{t_A(y) : y \in [x]_{R_\lambda} \cap [x]_{S_\lambda}\} \leqslant sup\{t_A(y) : y \in [x]_{R_\lambda}\} \wedge sup\{t_A(y) : y \in [x]_{S_\lambda}\}$$
$$= t_{\overline{A}_{R_\lambda}}(x) \wedge t_{\overline{A}_{S_\lambda}}(x),$$

$$f_{\overline{A}_{R_\lambda \cap S_\lambda}}(x) = inf\{f_A(y) : y \in [x]_{R_\lambda \cap S_\lambda}\} = inf\{f_A(y) : y \in [x]_{R_\lambda} \cap [x]_{S_\lambda}\} \geqslant inf\{f_A(y) : y \in [x]_{R_\lambda}\} \vee inf\{f_A(y) : y \in [x]_{S_\lambda}\}$$
$$= f_{\overline{A}_{R_\lambda}}(x) \vee f_{\overline{A}_{S_\lambda}}(x).$$

Similarly, by Definition 2.1, we know that the conclusions hold. □

Theorem 3.6. *Let A be a vague set of U and let R be a fuzzy equivalence relation on U. For all $\lambda, \mu \in [0,1]$, if $\lambda < \mu$, then the λ, μ-lower and λ, μ-upper approximations satisfy $\underline{A}_{R_\lambda} \subseteq \underline{A}_{R_\mu}$ and $\overline{A}_{R_\lambda} \supseteq \overline{A}_{R_\mu}$, respectively.*

Proof. From Lemma 2.2 and Theorem 3.4, the conclusions can be easily proved. □

The results of the Theorem 3.6 show that the numbers of elements contained in the λ-lower approximation gradually increase when the parameter λ increases from 0 to 1. Conversely, the numbers of elements contained in the λ-upper approximation gradually decrease.

4. Roughness measure of a vague set in fuzzy approximation space

The roughness measure of an ordinary set in the universe of discourse developed by Pawlak [6]. In 1996, Banerjee and Sankar [15] proposed a roughness measure of a fuzzy set A, which is expressed as

$$\rho_A^{\alpha\beta} = 1 - \frac{|\underline{A}_\alpha|}{|\overline{A}_\beta|},$$

where $0 < \beta \leqslant \alpha \leqslant 1$, $\underline{A}_\alpha = \{x \in U : \mu_{\underline{A}}(x) \geqslant \alpha\}$ and $\overline{A}_\beta = \{x \in U : \mu_{\overline{A}}(x) \geqslant \beta\}$, the notation $|\cdot|$ denotes the cardinality of the set.

Afterwards, the roughness measure of a vague set in Pawlak approximation space was introduced by Wang et al. [29], and many properties of roughness measure were discussed. In this section, we will consider a roughness measure of a vague set in fuzzy approximation space using the existing work, which is to be a generalization of the existing results.

Let A be a vague set, we will give the definition of $\alpha\beta$-level sets of the lower approximation $\underline{A}$ and upper approximation $\overline{A}$ in the fuzzy approximation space (U, R).

Definition 4.1. The $\alpha\beta$-level sets of $\underline{A}$ and $\overline{A}$ denoted by $\underline{A}_{\alpha\beta}$ and $\overline{A}_{\alpha\beta}$, respectively, are defined as

$$\underline{A}_{\alpha\beta} = \left\{x \in U : t_{\underline{A}}(x) \geqslant \alpha,\ f_{\underline{A}}(x) \leqslant \beta\right\}, \tag{9}$$
$$\overline{A}_{\alpha\beta} = \{x \in U : t_{\overline{A}}(x) \geqslant \alpha,\ f_{\overline{A}}(x) \leqslant \beta\}, \tag{10}$$

where $0 < \alpha,\ \beta < 1$ and $\alpha + \beta \leqslant 1$.

Now, we can introduce the roughness measure of a vague set in fuzzy approximation space.

Definition 4.2. A roughness measure $\rho_A^{\alpha\beta}$ of the vague set A of U with respect to the parameters α, β in the fuzzy approximation space (U, R), is defined as

$$\rho_A^{\alpha\beta} = 1 - \frac{|\underline{A}_{\alpha\beta}|}{|\overline{A}_{\alpha\beta}|}. \tag{11}$$

Especially, $\rho_A^{\alpha\beta} = 0$ when $|\overline{A}_{\alpha\beta}| = 0$.

Actually, if the fuzzy equivalence relation R becomes an equivalence relation, then the Definition 4.2 will reduce to the concept of the roughness measure, which defined by Wang et al. [29], in Pawlak approximation space. In addition, if the vague set A changes into a fuzzy set, the Definition 4.2 will reduce to the notion of a roughness measure of a fuzzy set, defined by Banerjee and Sankar [15], based on the analysis of the Definition 3.3.

Similar to Definition 4.1, for all $\lambda \in [0,1]$, we can introduce the definition of $\alpha\beta$-level sets of the λ-lower approximation $\underline{A}_{R_\lambda}$ and λ-upper approximation $\overline{A}_{R_\lambda}$ of the vague set A in the fuzzy approximation space (U, R).

Definition 4.3. The $\alpha\beta$-level sets of $\underline{A}_{R_\lambda}$ and $\overline{A}_{R_\lambda}$ denoted by $\underline{A}_{\alpha\beta}(\lambda)$ and $\overline{A}_{\alpha\beta}(\lambda)$, respectively, are defined as

$$\underline{A}_{\alpha\beta}(\lambda) = \left\{x \in U : t_{\underline{A}_{R_\lambda}}(x) \geqslant \alpha,\ f_{\underline{A}_{R_\lambda}}(x) \leqslant \beta\right\}, \tag{12}$$

$$\overline{A}_{\alpha\beta}(\lambda) = \left\{x \in U : t_{\overline{A}_{R_\lambda}}(x) \geqslant \alpha,\ f_{\overline{A}_{R_\lambda}}(x) \leqslant \beta\right\}, \tag{13}$$

where $0 < \alpha,\ \beta < 1$ and $\alpha + \beta \leqslant 1$.

Based on Definition 4.3, a roughness measure including the level λ of a vague set in fuzzy approximation space can be defined as follows:

Definition 4.4. A roughness measure including the level λ of the vague set A of U with respect to the parameters $\alpha,\ \beta$ in the fuzzy approximation space (U, R), is defined as

$$\rho_A^{\alpha\beta}(\lambda) = 1 - \frac{|\underline{A}_{\alpha\beta}(\lambda)|}{|\overline{A}_{\alpha\beta}(\lambda)|}, \quad \lambda \in [0,1]. \tag{14}$$

Especially, $\rho_A^{\alpha\beta}(\lambda) = 0$ when $|\overline{A}_{\alpha\beta}(\lambda)| = 0$.

In total, for a fixed level $\lambda \in [0,1]$, since R_λ is an equivalence relation on U, the roughness measure including the level λ becomes the roughness measure of a vague set in Pawlak approximation space.

Obviously, from Definitions 4.2 and 4.3, the roughness measure $\rho_A^{\alpha\beta}$ and the roughness measure $\rho_A^{\alpha\beta}(\lambda)$ including the level λ of the vague set A in fuzzy approximation space possess the same properties as the roughness measure in Pawlak approximation space stated in Ref. [29]. Consequently, the detailed description of these properties is omitted here.

Next, we give another property of roughness measure $\rho_A^{\alpha\beta}(\lambda)$ including the level λ.

Theorem 4.1. *Let A be a vague set of U and let R be a fuzzy equivalence relation on U. For all $\lambda, \mu \in [0,1]$ and $\lambda < \mu$, the roughness measure $\rho_A^{\alpha\beta}(\lambda)$, $\rho_A^{\alpha\beta}(\mu)$ including the levels λ, μ of the vague set A in the fuzzy approximation space (U, R) satisfy $\rho_A^{\alpha\beta}(\lambda) \geqslant \rho_A^{\alpha\beta}(\mu)$.*

Proof. By Theorem 3.6 and Definition 4.3, we know that $\underline{A}_{\alpha\beta}(\lambda) \subseteq \underline{A}_{\alpha\beta}(\mu)$ and $\overline{A}_{\alpha\beta}(\lambda) \supseteq \overline{A}_{\alpha\beta}(\mu)$. According to Definition 4.4, we can easily obtain $\rho_A^{\alpha\beta}(\lambda) \geqslant \rho_A^{\alpha\beta}(\mu)$. □

The result shows that the roughness measure including the level λ of vague sets decreases as the level λ gradually becomes bigger in [0,1].

5. An illustrative example

In this section, we will illustrate how to calculate the lower and upper approximations of a vague set of the finite and nonempty universe, in fuzzy approximation space, with the β-product between two fuzzy matrixes. And in further obtain the calculation of the λ-lower and λ-upper approximations of a vague set. Besides, the roughness measure and that including the level λ of a vague set can also be obtained. Let us consider the following example. Let $U = \{x_1, x_2, \ldots, x_5\},\ R$ is a fuzzy equivalence relation on U, which can also be denoted by the fuzzy equivalence matrix, as follows:

$$R(x_i, x_j) = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{bmatrix} \quad (i,j = 1,2,\ldots,5).$$

Let A be a vague set and $A = \{[0.2, 0.5]/x_1, [0.7, 0.8]/x_2, [0.4, 0.6]/x_3, [0.1, 0.4]/x_4, [0.6, 0.9]/x_5\}$.

Notice that the expression of vague sets is given by the following form, namely

$$A = \{[t_A(x), 1 - f_A(x)]/x : x \in U\}.$$

The truth-membership function and false-membership function of the vague set A are viewed as two column vectors, i.e., two 5×1-order matrixes. And they are expressed as the t_A and f_A, respectively. Consequently, we have

$$t_A = (t_A(x_1), t_A(x_2), t_A(x_3), t_A(x_4), t_A(x_5))^T = (0.2, 0.7, 0.4, 0.1, 0.6)^T,$$
$$f_A = (f_A(x_1), f_A(x_2), f_A(x_3), f_A(x_4), f_A(x_5))^T = (0.5, 0.2, 0.4, 0.6, 0.1)^T.$$

By Definitions 2.5, 2.6 and 3.1, the lower and upper approximations of the vague set A can be obtained as follows:

$$t_{\underline{A}} = R\beta t_A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{bmatrix} \beta \begin{bmatrix} 0.2 \\ 0.7 \\ 0.4 \\ 0.1 \\ 0.6 \end{bmatrix} = (0.2, 0.25, 0.25, 0.1, 0.25)^T,$$

$$t_{\overline{A}} = R\beta^* t_A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{bmatrix} \beta^* \begin{bmatrix} 0.2 \\ 0.7 \\ 0.4 \\ 0.1 \\ 0.6 \end{bmatrix} = (0.625, 0.7, 0.647, 0.1, 0.647)^T,$$

$$f_{\underline{A}} = R\beta^* f_A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{bmatrix} \beta^* \begin{bmatrix} 0.5 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.1 \end{bmatrix} = (0.5, 0.375, 0.4, 0.6, 0.375)^T,$$

$$f_{\overline{A}} = R\beta f_A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{bmatrix} \beta \begin{bmatrix} 0.5 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.1 \end{bmatrix} = (0.125, 0.118, 0.111, 0.5, 0.1)^T.$$

Hence, we have

$$\underline{A} = \{[0.2, 0.5]/x_1, [0.25, 0.625]/x_2, [0.25, 0.6]/x_3, [0.1, 0.4]/x_4, [0.25, 0.625]/x_5\},$$
$$\overline{A} = \{[0.625, 0.875]/x_1, [0.7, 0.882]/x_2, [0.647, 0.889]/x_3, [0.1, 0.5]/x_4, [0.647, 0.9]/x_5\}.$$

According to Definition 4.1, if $\alpha = 0.2,\ \beta = 0.4$, then the $\alpha\beta$-level sets of $\underline{A}$ and $\overline{A}$ are calculated, respectively, as follows:

$$\underline{A}_{\alpha\beta} = \{x_2, x_3, x_5\}, \quad \overline{A}_{\alpha\beta} = \{x_1, x_2, x_3, x_5\}.$$

Therefore, the roughness measure of the vague set A can be obtained as $\rho_A^{\alpha\beta} = 1 - \frac{|\underline{A}_{\alpha\beta}|}{|\overline{A}_{\alpha\beta}|} = 1 - \frac{3}{4} = 0.25$.

In addition, for all $\lambda \in [0, 1]$, because λ-cut matrix is an equivalence matrix, the universe U can be partitioned by the equivalence matrix R_λ. Obviously, when λ takes the different values in $[0, 1]$, the universe U will be divided into the different equivalence classes. The detailed classifications are listed as follows:

(c1) If $0 \leqslant \lambda \leqslant 0.2$, there is one class: $\{x_1, x_2, x_3, x_4, x_5\}$;
(c2) If $0.2 < \lambda \leqslant 0.8$, there are two classes: $\{x_1, x_2, x_3, x_5\},\ \{x_4\}$;
(c3) If $0.8 < \lambda \leqslant 0.85$, there are three classes: $\{x_1\},\ \{x_2, x_3, x_5\},\ \{x_4\}$;
(c4) If $0.85 < \lambda \leqslant 0.9$, there are four classes: $\{x_1\},\ \{x_2\},\ \{x_3, x_5\},\ \{x_4\}$;
(c5) If $0.9 < \lambda \leqslant 1$, there are five classes: $\{x_1\},\ \{x_2\},\ \{x_3\},\ \{x_4\},\ \{x_5\}$.

Next, we consider the λ-lower and λ-upper approximations of the vague set A. If $\lambda = 0.6,\ \mu = 0.85$, we have

$$\underline{A}_{R_\lambda} = \{[0.2, 0.5]/x_1, [0.2, 0.5]/x_2, [0.2, 0.5]/x_3, [0.1, 0.4]/x_4, [0.2, 0.5]/x_5\},$$
$$\overline{A}_{R_\lambda} = \{[0.7, 0.9]/x_1, [0.7, 0.9]/x_2, [0.7, 0.9]/x_3, [0.1, 0.4]/x_4, [0.7, 0.9]/x_5\},$$
$$\underline{A}_{R_\mu} = \{[0.2, 0.5]/x_1, [0.4, 0.6]/x_2, [0.4, 0.6]/x_3, [0.1, 0.4]/x_4, [0.4, 0.6]/x_5\},$$
$$\overline{A}_{R_\mu} = \{[0.2, 0.5]/x_1, [0.7, 0.9]/x_2, [0.7, 0.9]/x_3, [0.1, 0.4]/x_4, [0.7, 0.9]/x_5\}.$$

By Definition 4.3, if $\alpha = 0.3,\ \beta = 0.6$, then the $\alpha\beta$-level sets of $\underline{A}_{R_\lambda},\ \underline{A}_{R_\mu}$ and $\overline{A}_{R_\lambda},\ \overline{A}_{R_\mu}$ are calculated, respectively, as follows:

$$\underline{A}_{\alpha\beta}(\lambda) = \emptyset, \quad \overline{A}_{\alpha\beta}(\lambda) = \{x_1, x_2, x_3, x_5\};$$
$$\underline{A}_{\alpha\beta}(\mu) = \{x_2, x_3, x_5\}, \quad \overline{A}_{\alpha\beta}(\mu) = \{x_2, x_3, x_5\}.$$

Similarly, the roughness measures including the levels λ and μ of the vague set A can be obtained, respectively.

$$\rho_A^{\alpha\beta}(\lambda) = 1 - \frac{|\underline{A}_{\alpha\beta}(\lambda)|}{|\overline{A}_{\alpha\beta}(\lambda)|} = 1 - \frac{0}{4} = 1,$$
$$\rho_A^{\alpha\beta}(\mu) = 1 - \frac{|\underline{A}_{\alpha\beta}(\mu)|}{|\overline{A}_{\alpha\beta}(\mu)|} = 1 - \frac{3}{3} = 0.$$

6. Conclusions

The integration of the vague set theory and rough set theory is an interesting and valuable research work, which has been studying until now. In 2005, the rough approximations of a vague set in Pawlak approximation space were proposed by Wang, etc. However, a problem was presented, which was how to extend the rough approximations of a vague set to the fuzzy approximation space. Therefore, we investigate the rough approximations of the vague sets in fuzzy approximation space to give some answers for the previous problem.

In this paper, we introduce the definition of the lower and upper approximations of a vague set in fuzzy approximation space based on the β-operator and β-complement operator. In fact, the concept is not only an extension of the fuzzy rough sets, but also can be regarded as a generalization of the rough vague sets defined in Pawlak approximation space. The rough approximations of a vague set are different from the ones of a fuzzy set. The former is not only consider the meaning of approximation operators, but also take into account the conditions required by both the truth and false-membership functions of approximation operators. While the latter is not the case. In addition, λ-lower and λ-upper approximations containing the parameter λ are also proposed. In essence, these approximation operators are rough approximation operators defined in Pawlak approximation space. Since that some properties of these two types are studied, and show that the lower and upper approximations of vague sets in fuzzy approximation space have the same properties as those in the classical rough sets, fuzzy rough sets, rough fuzzy sets and vague sets defined in Pawlak approximation space. Similarly, we have also defined a roughness measure of a vague set in fuzzy approximation space.

In the past, most approximation operators of a fuzzy or vague set have always been defined by the operators "$\vee$" and "$\wedge$" (or "sup" and "inf") in any approximation space. Whereas, those operators are so limited that it is difficult to accurately unearth the essence of fuzzy phenomenon. Hence, it is necessary to define rough approximation operators by different methods.

To sum up, this paper is a new combination of the rough set theory, vague set theory and fuzzy set theory, and will help deal with some incomplete and imprecise issues. Of course, it will also bring us with many new problems worth considering.

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